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Weyl Geometry and Quantum Gravity

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Abstract

It is shown that the recently geometric formulation of quantum mechanics

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[1–6] implies the use of Weyl geometry. It is discussed that the natural framework for both gravity and quantum is Weyl geometry. At the end a Weyl invariant theory is built, and it is shown that both gravity and quantum are present at the level of equations of motion. The theory is applied to cosmology leading to the desired time dependencies of the cosmological and gravitational constants.

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I. INTRODUCTION AND SURVEY

Why Weyl geometry? From a long time ago it is believed that the long range forces (i.e. electromagnetism and gravity) are different aspects of a unique phenomena. So they must be unified. Usually it is proposed that one must generalize Einstein's general relativity theory to have a geometrical description of electromagnetic fields. This means to change the properties of the manifold of general relativity. Using higher dimensional manifolds [7,8], changing the compatibility relation between the metric and the affine connection [9,10] and using a non-symmetric metric [11,12] are some examples of the attempts towards this idea. In all the above approaches, the additional degrees of freedom correspond to the components of the electromagnetic potential.

The second idea leads to the Weyl's gauge invariant geometry. In Weyl geometry, both the components and the length of a vector change linearly proportional to the infinitesimal translation during any parallel transportation. The former produces some rotation of the vector, as in Riemanian geometry, while the latter is a special aspect of Weyl geometry expressed as:

$$\delta\ell = \phi_\mu \delta x^\mu \ell; \qquad or \quad \ell = \ell_0 \exp\left(\int \phi_\mu dx^\mu\right) \tag{1}$$

So the length unit is different at different points. Relation (1) presents a new vector field (ϕ_μ) in the theory, identified with the electromagnetic potential by Weyl [9,10]. This equation shows that the change of length between two arbitrary points is dependent on the chosen path unless the curl of the vector field is zero (non–integrability of lenght).

Equivalently, the length change can be replaced by a change in the metric as:

$$g_{\mu\nu} \to \exp\left(2\int \phi_\mu dx^\mu\right) g_{\mu\nu} \tag{2}$$

which is called Weyl transformation. We see that the metric is a Weyl covariant (or co-covariant) object of the weight 2. Now assumming that contravariant vectors change during any parrallel transportation as in Riemanian geometry, relation (1) shows that the Weyl affine connection is given by:

$$ {}^{W}\Gamma^{\mu}_{\nu\lambda} = \left\{ \begin{array}{c} \mu \\ \nu\lambda \end{array} \right\} + g_{\nu\lambda}\phi^{\mu} - \delta^{\mu}_{\nu}\phi_{\lambda} - \delta^{\mu}_{\lambda}\phi_{\nu} \tag{3} $$

differring from Christoffel symbols by the last three terms.

Now suppose that we make the following transformation:

$$ \phi_{\mu} \longrightarrow \phi'_{\mu} = \phi_{\mu} + \partial_{\mu}\Lambda \tag{4} $$

which is called a *gauge transformation*. The effect of this is to transform $g_{\mu\nu} \rightarrow g'_{\mu\nu} = \exp(2\Lambda)g_{\mu\nu}$ and $\delta\ell \rightarrow \delta\ell' = \delta\ell + (\partial_{\mu}\Lambda)\delta x^{\mu}\ell$. One can see the change in lenght is the same in the two gauges when one turns around a closed path. So a gauge transformation does not change the geometry.

The quantity defined as the curl of the Weyl vector:

$$ F_{\mu\nu} = \partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu} \tag{5} $$

is gauge invariant, and corresponds to the electromagnetic field. If the Weyl vector be the gradient of a scalar, there exists a gauge transformation leading to a zero Weyl vector field or equivalently Riemannian geometry class. In this case the lenght is integrable. So the properties of the elementary particles are independent of their path history.

Appart from the electromagnetic aspects of Weyl geometry, it has some other applications. Some authors believe that Weyl geometry is a suitable framework for quantum gravity. In ref [13] a new quantum theory is proposed on the basis of Weyl picture which is

purely geometric. The observables are introduced as zero Weyl weight quantities. Moreover any weightful field has a Weyl conjugate such as complex conjugate of the state vector in quantum mechanics. By these dual fields, the probability can be defined. These are the elements of a consistent quantum theory which is equivalent to the standard quantum mechanics. Moreover it is shown that the quantum measurement and the related uncertainty would emerged from Weyl geometry naturally. In this theory when the curl of Weyl vector is zero, we arrive at the classical limit. By noting the relation(4), it is concluded that the change of length scale is only a quantum effect.

Another different approach to geometrize quantum mechanics can be found in [14]. Here a modified Weyl–Dirac theory is used to join the particle aspects of matter and Weyl symmetry breaking.

In the present work we shall look at the conformal invariance at the quantum level. Does the quantum theory lead us to any characteristic length scale and thus break the conformal symmetry? Or conversely the quantum effects lead us to a conformal invariant geometry? We shall discuss these questions in the context of the causal quantum theory proposed by Bohm [15–17].

This paper is organized in the following manner. In section II we shall discuss the Weyl–Dirac theory in details. One of the main points of this paper comes in section III where we shall show that the Weyl vector and the quantum effects of matter are connected, so answering the question why Weyl geometry. In this sense it may be suitable to name Weyl geometry as quantum geometry. We shall preciesly show in this section how the conformal symmetry emerges naturally by considering quantum effects of matter. Finally in section IV we show that the Weyl–Dirac theory is a suitable framwork for identification of the

conformal degree of freedom of the space–time with the Bohm's quantum mass.

II. WEYL–DIRAC THEORY

Straightforward generalization of Einstein–Hilbert action to Weyl geometry leads to a higher order theory [9,10]. Dirac [18,19] introduced a new action called Weyl–Dirac action, by including a new field which is in fact gauge function. It helps him to avoid higher order actions as while as fixing the gauge function leads to Einstein–Maxwell equations.

The Weyl–Dirac action is given by [18,19]:

$$\mathcal{A} = \int d^4x\sqrt{-g}\left(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\!\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathcal{L}_{matter}\right) \tag{6}$$

where β is a scalar field of weight -1. The ";" represents covariant derivation under general coordinate and conformal transformations (Weyl covariant derivative) defined as:

$$X_{;\mu} = {}^W\!\nabla_\mu X - \mathcal{N}\phi_\mu X \tag{7}$$

where $\mathcal{N}$ is the Weyl weight of X. The equations of motion then would be:

$$\mathcal{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathcal{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}\ {}^W\!\nabla^\alpha\ {}^W\!\nabla_\alpha\beta - {}^W\!\nabla^\mu\ {}^W\!\nabla^\nu\beta)$$

$$+\frac{1}{\beta^2}(4\nabla^\mu\beta\nabla^\nu\beta - g^{\mu\nu}\nabla^\alpha\beta\nabla_\alpha\beta) + \frac{\sigma}{\beta^2}(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha}) \tag{8}$$

$${}^W\!\nabla_\nu F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^\mu + \beta\nabla^\mu\beta) + 4\pi J^\mu \tag{9}$$

$$\mathcal{R} = -(\sigma+6)\frac{{}^W\!\Box\beta}{\beta} + \sigma\phi_\alpha\phi^\alpha - \sigma\ {}^W\!\nabla^\alpha\phi_\alpha + \frac{\psi}{2\beta} \tag{10}$$

where:

$$M^{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu}_{\alpha}F^{\nu\alpha}\right) \tag{11}$$

and the energy–momentum tensor $\mathcal{T}^{\mu\nu}$, current density vector J^{μ} and the scalar ψ are defined as:

$$8\pi\mathcal{T}^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{matter}}{\delta g_{\mu\nu}} \tag{12}$$

$$16\pi J^{\mu} = \frac{\delta\mathcal{L}_{matter}}{\delta\phi_{\mu}} \tag{13}$$

$$\psi = \frac{\delta\mathcal{L}_{matter}}{\delta\beta} \tag{14}$$

On the other hand the equation of motion of matter and the trace of energy-momentum tensor can be resulted from the invariance of action under the coordinate and gauge transformations. One can write them as respectively:

$$^{W}\nabla_{\nu}\mathcal{T}^{\mu\nu} - \mathcal{T}\frac{\nabla^{\mu}\beta}{\beta} = J_{\alpha}\phi^{\alpha\mu} - (\phi^{\mu} + \frac{\nabla^{\mu}\beta}{\beta})\ {}^{W}\nabla_{\alpha}J^{\alpha} \tag{15}$$

$$16\pi\mathcal{T} - 16\pi\ {}^{W}\nabla_{\mu}J^{\mu} - \beta\psi = 0 \tag{16}$$

The first of them is only a geometrical identity (Bianchi identity) and the second results from the non independence of field equations.

It must be noted that in the Weyl–Dirac theory, the Weyl vector does not couple to spinors, so ϕ_{μ} cannot be interpretad as the electromagnetic potential [20]. Here we use the Weyl vector not as the electromagnetic field but only as a part of the geometry of the space–time. The Weyl–Dirac formalism is adopted and we shall see that the auxiliary field (gauge function) in Dirac's action represents the quantum mass field. In addition both gravitation fields ($g_{\mu\nu}$ and ϕ_{μ}) and quantum mass field determine the geometry of the space–time.

III. BOHMIAN QUANTUM GRAVITY AND WEYL SYMMETRY

In a series of papers [1–6] we have proposed a new approach to quantum gravity based on the de-Broglie–Bohm quantum theory. The approach in the above references is different from the standard Bohmian quantum gravity presented e.g. in [21]. These works are attempts to geometrize the quantum behaviour of matter. This point, as a conjecture, firstly proposed by de-Broglie [22] which states that the quantum effects can be removed via a conformal transformation. We have shown in [1] that the quantum effects can be included in the conformal degree of freedom of the space–time metric. Adding the gravitational effects, it can be seen that quantum and gravity are highly coupled. This produces remarkable changes in the classical predictions such as the physics of the birth of the universe as long as the classical limit is obtained correctly.

One of the new points of the above approach is the dual role of geometry in physics. The gravitational effects determine the causal structure of the space–time as long as the quantum effects give its conformal structure. This does not mean that the quantum effects has nothing to do with the causal structure, it can act on the causal structure through back–reaction terms appeared in the metric field equations [2,3,6]. We only mean that the dominant term in the causal structure is the gravitational effects. The same is true for the conformal factor. The conformal factor of metric is a function of quantum potential which is the principal character in Bohm's theory and given by:

$$\mathcal{Q} = \alpha \frac{\Box\sqrt{\rho}}{\sqrt{\rho}}; \qquad \alpha = \frac{\hbar^2}{m^2c^2} \tag{17}$$

where ρ is the ensemble density of the system. According to Bohm the mass of a relativistic particle is a field produced by quantum corrections to classical mass. This can be easily seen

from the quantum-Hamilton-Jacobi equation for a spin-less particle:

$$\nabla_\mu S \nabla^\mu S = \mathcal{M}^2 c^2 = m^2 c^2 (1 + \mathcal{Q}) \tag{18}$$

where S is the Hamilton–Jacobi function and $\mathcal{M}$ is the quantum mass. We have shown in ref [2,3,6] that the presence of quantum potential is equivalent to a conformal mapping of the metric. Thus in the conformally related frames we feel different quantum masses and different curvatures correspondingly. It is possible to consider two specific frames. One of them contains the quantum mass field (appeared in quantum Hamilton-Jacobi equation) and the classical metric as long as in the other the classical mass (appeared in classical Hamilton-Jacobi equation) and quantum metric are appeared. In other frames both the space–time metric and mass field contain the quantum properties. This argument motivates us to state that *different conformal frames are identical pictures of the gravitational and quantum phenomena*. Considering the quantum force, the conformally related frames aren't distinguishable. This is just what happens when we consider gravity, different coordinate systems are equivalent. Since the conformal transformation change the length scale locally, therefore we feel different quantum forces in different conformal frames. This is similar to general relativity in which general coordinate transformation changes the gravitational force at any arbitrary point. Here it may be appropriate to state a basic question. Does applying the above correspondence, between quantum and gravitational forces, and between the conformal and general coordinate transformations, means that the geometrization of quantum effects implies the conformal invariance as gravitational effects imply the general coordinate invariance?

In order to discuss this question, we recall what were considered early in the development

of the theory of general relativity. General covariance principle leads to the identification of gravitational effects of matter with the geometry of the space–time. In general relativity the important fact which supports this identification is the equivalence principle. According to it, one can always remove the gravitational field at some point by a suitable coordinate transformation. Similarly, as we pointed out previously, according to our new approach to Bohmian quantum gravity, at any point (or even globally) the quantum effects of matter can be removed by a suitable conformal transformation. Thus in that point(s) matter behaves classically. In this way we can introduce a new equivalence principle calling it as *conformal equivalence principle* similar to the standard equivalence principle [5]. The latter interconnects gravity and general covariance while the former has the same role about quantum and conformal covariance. Both of these principles state that there isn't any preferred frame, either coordinate or conformal frame. Since Weyl geometry welcomes conformal invariance and since it has additional degrees of freedom which can be identified with quantum effects, it provides a unified geometrical framework for understanding the gravitational and quantum forces. In this way a pure geometric interpretation of quantum behavior can be built.

Because of these results, we believe that the de-Broglie–Bohm theory must receive increasing attention in quantum gravity. This theory has some important features. One of them is that the quantum effects appear independent of any preferred scale length (this is opposite to the standard quantum mechanics in which Plank length is the characteristic length). This is one of the intrinsic properties of this theory which resulted from the special definition of the classical limit [21]. Another important aspect is that the quantum mass of the particle is a field. This is needed for having conformal invariance, since mass has a

non–zero Weyl weight. Also as we have shown previously [2,3,6] the guiding equation lead us to the following geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \frac{1}{\mathcal{M}} \left(g^{\mu\nu} - \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \right) \nabla_\nu \mathcal{M} \tag{19}$$

The appearance of quantum mass justifies the Mach's principle [23] which leads to the existence of interrelation between global properties of the universe (space–time structure, the large scale structure of the universe,$\cdots$) and its local properties (local curvature, motion in a local frame,$\cdots$). In the present theory, it can be easily seen that the geometry of the space–time is determined by the distribution of matter. A local variation of matter field distribution changes the quantum potential acting on geometry. Thus the geometry would be altered globally (in conformation with Mach's principle). In this sense our approach to the quantum gravity is highly non–local as it is forced by the nature of the quantum potential [4]. What we call geometry is only the gravitational and quantum effects of matter. Without matter the geometry would be meaningless. Moreover in [2,6] we have shown that it is necessary to assume an interaction term between the cosmological constant (large scale structure) and the quantum potential (local phenomena). These properties all justify Mach's principle. It is shown in [2,6] that the gravitational constant is in fact a field depending on the matter distribution through quantum potential.

All these arguments based on Bohmian quantum mechanics motivates us that the Weyl geometry is a suitable framework for formulating quantum gravity.

IV. WEYL INVARIANT QUANTUM GRAVITY

In this section we shall construct a theory for Bohmian quantum gravity which is conformal invariant in the framework of Weyl geometry. To begin, note that if our model should consider massive particles, the mass must be a field. This is because mass has non–zero Weyl weight. This is in agreement with Bohm's theory. As we argued previousely a general Weyl invariant action is the Weyl–Dirac action, whose equations of motion are derived in section II. To simplify our model, we assume that the matter lagrangian does not depends on the Weyl vector, so that $J_\mu = 0$. The equations of motion are now:

$$\mathcal{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathcal{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}\ {}^W\!\nabla^\alpha\ {}^W\!\nabla_\alpha\beta - \ {}^W\!\nabla^\mu\ {}^W\!\nabla^\nu\beta)$$

$$+\frac{1}{\beta^2}(4\nabla^\mu\beta\nabla^\nu\beta - g^{\mu\nu}\nabla^\alpha\beta\nabla_\alpha\beta) + \frac{\sigma}{\beta^2}(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha}) \tag{20}$$

$${}^W\!\nabla_\nu F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^\mu + \beta\nabla^\mu\beta) \tag{21}$$

$$\mathcal{R} = -(\sigma+6)\frac{{}^W\!\Box\beta}{\beta} + \sigma\phi_\alpha\phi^\alpha - \sigma\ {}^W\!\nabla^\alpha\phi_\alpha + \frac{\psi}{2\beta} \tag{22}$$

and symmetry conditions are:

$${}^W\!\nabla_\nu\mathcal{T}^{\mu\nu} - \mathcal{T}\frac{\nabla^\mu\beta}{\beta} = 0 \tag{23}$$

$$16\pi\mathcal{T} - \beta\psi = 0 \tag{24}$$

It must be noted that from equation (21) we have:

$${}^W\!\nabla_\mu\left(\beta^2\phi^\mu + \beta\nabla^\mu\beta\right) = 0 \tag{25}$$

so ϕ_μ is not independent of β.

It is worthwhile to see whether this model has anything to do with the Bohmian quantum gravity or not. We want to introduce the quantum mass field. Now we shall show that this field is proportional to the Dirac field. In order to see this two conditions are neccessary to meet. Firstly the correct dependence of Dirac field on the trace of energy–momentum tensor and secondly the correct appearance of quantum force in the geodesic equation. Now note that using equations (21),(22), and (24) we have:

$$\Box\beta + \frac{1}{6}\beta\mathcal{R} = \frac{4\pi}{3}\frac{\mathcal{T}}{\beta} + \sigma\beta\phi_\alpha\phi^\alpha + 2(\sigma-6)\phi^\gamma\nabla_\gamma\beta + \frac{\sigma}{\beta}\nabla^\mu\beta\nabla_\mu\beta \tag{26}$$

This equation can be solved iteratively. Let we rewrite it as:

$$\beta^2 = \frac{8\pi\mathcal{T}}{\mathcal{R}} - \frac{1}{\mathcal{R}/6 - \sigma\phi_\alpha\phi^\alpha}\beta\Box\beta + \cdots \tag{27}$$

The first and the second order solutions of this equation is:

$$\beta^{2(1)} = \frac{8\pi\mathcal{T}}{\mathcal{R}} \tag{28}$$

$$\beta^{2(2)} = \frac{8\pi\mathcal{T}}{\mathcal{R}}\left(1 - \frac{1}{\mathcal{R}/6 - \sigma\phi_\alpha\phi^\alpha}\frac{\Box\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} + \cdots\right) \tag{29}$$

In order to derive the geodesic equation we use the relation (23). Assuming that matter is consisted of dust with the energy–momentum tensor:

$$\mathcal{T}^{\mu\nu} = \rho u^\mu u^\nu \tag{30}$$

where ρ and u^μ are matter density and velocity respectively, substituting (30) into (23) and multiplying by u_μ, gives us:

$${}^{W}\!\nabla_\nu(\rho u^\nu) - \rho\frac{u_\mu\nabla^\mu\beta}{\beta} = 0 \tag{31}$$

If we substitute (23) into (31) again, we have:

$$u^\nu\ {}^W\!\nabla_\nu u^\mu = \frac{1}{\beta}(g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\beta \tag{32}$$

Comparison of equations (29) and (32) with equations (18) and (19) shows that we have the coorect equations of motions of Bohmian quantum gravity, provided we identify:

$$\beta \longrightarrow \mathcal{M} \tag{33}$$

$$\frac{8\pi\mathcal{T}}{\mathcal{R}} \longrightarrow m^2 \tag{34}$$

$$\frac{1}{\sigma\phi_\alpha\phi^\alpha - \mathcal{R}/6} \longrightarrow \alpha \tag{35}$$

V. APPLICATION TO COSMOLOGY

Most of physicists believe in a non–zero cosmological constant because of two important reasons. It helps us to make the theoretical results to agree with observations. Morever some topics, like large scale structure of the universe, dark matter, inflation, can be explored using it. On the other hand from astronomical observations, especially gravitational lensing, cosmological constant should be very small. ($|\Lambda| < 10^{-54}/cm^2$) The fact that the cosmological constant is small produces some difficulties. How explain theoretically this value of the cosmological constant? (This question also applies to the gravitation coupling constant.) Morever the cosmological constant is a measure of vaccum energy density. This includes some contribution from scalar fields, bare cosmological constant, quantum effect, and so on. But observed cosmological constant is more smaller than (120 order of magnitude less than) each one of the above contributions. This is the so–called cosmological constant

puzzle (see [24] and its references). Till now many mechanisms are presented to solve the problem.

One way to solve the problem is to give dynamical characters to gravitational and cosmological constants in such a way that they decrease as the universe expands. Some works are done in [25] and [26]. In the former, a mechanism is presented using the WDW equation, while the latter, focuses on the breaking the conformal invariance. Two scales, cosmological and particle physics are introduced. And a dynamical conformal factor which relates them produces an effective time dependent cosmological constant.

We also use the conformal invariance, but in the conformal invariant framework of the present paper. Let's choose a spatially flat Robertson–Walker metric:

$$ds^2 = a^2(\eta)\left[d\eta^2 - dr^2 - r^2 d\Omega^2\right] \tag{36}$$

where $a(\eta)$ is the scale factor, and assuming the universe is filled of a dust, the equations of motion of theory presented in the previous section now simplifies to:

$$3\frac{\dot{a}^2}{a^4} - \frac{8\pi\rho}{\beta^2} + \frac{6}{\beta}\left(\frac{\dot{a}}{a} - \phi\right)\frac{\dot{\beta}}{a^2} + \frac{3}{\beta^2}\frac{\dot{\beta}^2}{a^2} + \frac{\sigma}{2\beta^2}\frac{(\dot{\beta}+\phi\beta)^2}{a^2} = 0 \tag{37}$$

$$\dot{\beta} + \beta\phi = 0 \tag{38}$$

$$-6\frac{\ddot{a}}{a^3} - (\sigma+6)\left(\frac{1}{\beta}\frac{d}{d\eta}\left(\frac{\dot{\beta}}{a^2}\right) + \frac{\dot{\beta}}{\beta a^2}\left(4\frac{\dot{a}}{a} - 10\phi\right)\right) + \sigma\frac{\phi^2}{a^2} - \sigma\frac{d}{d\eta}\left(\frac{\phi}{a^2}\right) - \sigma\frac{\phi}{a^2}\left(4\frac{\dot{a}}{a} - 10\phi\right) + \frac{\psi}{2\beta} = 0 \tag{39}$$

where a dot over any quantity represents derivation with respect to time and we have chosen the gauge

$$\phi_\mu = (\phi, 0, 0, 0) \tag{40}$$

And the symmetry conditions are:

$$\dot{\rho} + 3\rho\left(\frac{\dot{a}}{a} - \phi\right) - \rho\frac{\dot{\beta}}{\beta} = 0 \tag{41}$$

$$16\phi\rho - \beta\psi = 0 \tag{42}$$

Introducing the cosmological time as $dt = ad\eta$ and simplifying the relations, we finally have:

$$\rho a^3\beta^2 = constant \tag{43}$$

$$3\frac{a'^2}{a^2} - \Lambda_{eff} - 8\pi G_{eff}\rho = 0 \tag{44}$$

$$3\frac{a''}{a} + 3\frac{a'^2}{a^2} + 30\frac{\beta'^2}{\beta^2} + 9\frac{a'}{a}\frac{\beta'}{\beta} + 3\frac{\beta''}{\beta} - 4\pi G_{eff}\rho = 0 \tag{45}$$

where a $'$ over any quantity represents derivation with respect to the cosmological time and we have deffined:

$$\Lambda_{eff} = -9\frac{\beta'^2}{\beta^2} - 6\frac{a'}{a}\frac{\beta'}{\beta} \tag{46}$$

$$G_{eff} = \frac{1}{\beta^2} \tag{47}$$

The above equations can simply solved resulting in:

$$H \sim t^{-1} \tag{48}$$

$$\Lambda_{eff} \sim t^{-2} \tag{49}$$

$$G_{eff} \sim t^{-4/19} \tag{50}$$

where H is the Hubble constant. As the universe expands these quantities decrease in agreement with the above disscusion. These constants have a small value at the current epoch as the observation suggests.

VI. CONCLUSION

We addressed the question "*Why Weyl geometry?*". Among all the arguments in the favor of it, the most important one is that the conformal degree of freedom of the space–time metric, should be identified with Bohm's quantum potential. We saw that one can formulate a *generalized equivalence principle* which states that gravitation can be removed locally via an appropriate coordinate transformation, while quantum force can be removed either locally or globally via an appropriate scale transformation. So the natural framework of quantum and gravity is Weyl geometry. The most simplest Weyl invariant action functional is written out. It surprisingly leads to the correct Bohm's equations of motion. When it applied to cosmology it leads to time decreasing cosmological and gravitational constants. A phenomena which is good for describing their small values.

Since a gauge transformation can transform a general space–time dependent Dirac field to a constant one, and vice-versa, it can be shown that quantum effects and the lenght scale of the space–time are closely related. To see this suppose we are in a gauge in which Dirac field is a constant. By applying a gauge transformation one can change it to a general space–time dependent function.

$$\beta = \beta_0 \longrightarrow \beta = \beta(x) = \beta_0 \exp(-\Lambda(x)) \tag{51}$$

This gauge transformation is defined as:

$$\phi_\mu \longrightarrow \phi_\mu + \partial_\mu \Lambda \tag{52}$$

So, the gauge in which the quantum mass is constant (and thus the quantum force is zero) and the gauge in which the quantum mass is space–time dependent are related to each other

via a scale change. In other words, ϕ_μ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$. Since ϕ_μ is a part of Weyl geometry, and Dirac field represents the quantum mass, one concludes that the quantum effects are geometrized. One can see this fact also by referring to the equation (25) which shows that ϕ_μ is not independent of β, so the Weyl vector is determined by quantum mass, and thus this geometrical aspect of the manifold is related to the quantum effects. In this way, the physical meaning of auxiliary Dirac field is clarified, as while as a suitable model for quantum gravity is introduced.

In a forthcoming paper we shall investigate the solutions of the field equations of this theory.

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[25] T. Horiguchi, *Il Nuovo Cimento*, 112**B**, No. 9, 1227, 1997.

[26] Y. Bisabr, and H. Salehi, Class. Quant. Grav., **19**, 2369, 2002.